4. Quantum Phase Estimation

Isaac H. Kim (UC Davis)

Quantum Phase Estimation

- Quantum Phase Estimation (QPE) is one of the most widely used subroutines in quantum algorithms.
- Setup: We have a unitary U and its eigenstate $|n\rangle$, with an eigenvalue $e^{i\theta_n}$.
- QPE is an algorithm to measure θ_n.
- QPE serves two purposes in Quantum Simulation.
  - Estimating ground state energy.
  - Ground state preparation

$U|n\rangle = e^{i\theta_n}|n\rangle$

How QPE works

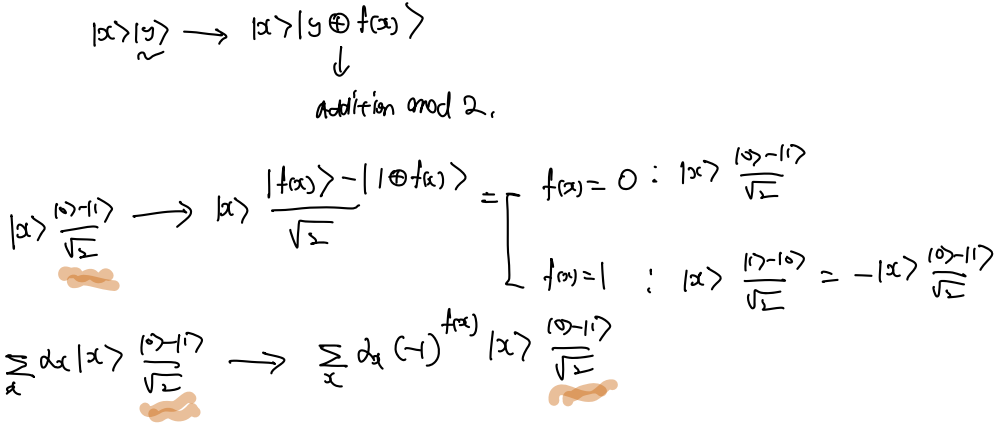
- There are two ingredients.
    1. Phase kick-back
    2. Quantum Fourier Transform

Phase kickback

- Simplest case: Suppose we have an ability to compute a function $f(x) \in \{0,1\}$. We wish to implement a unitary transformation:

$$|x\rangle \rightarrow (-1)^{f(x)}|x\rangle .$$

- How would we do this?



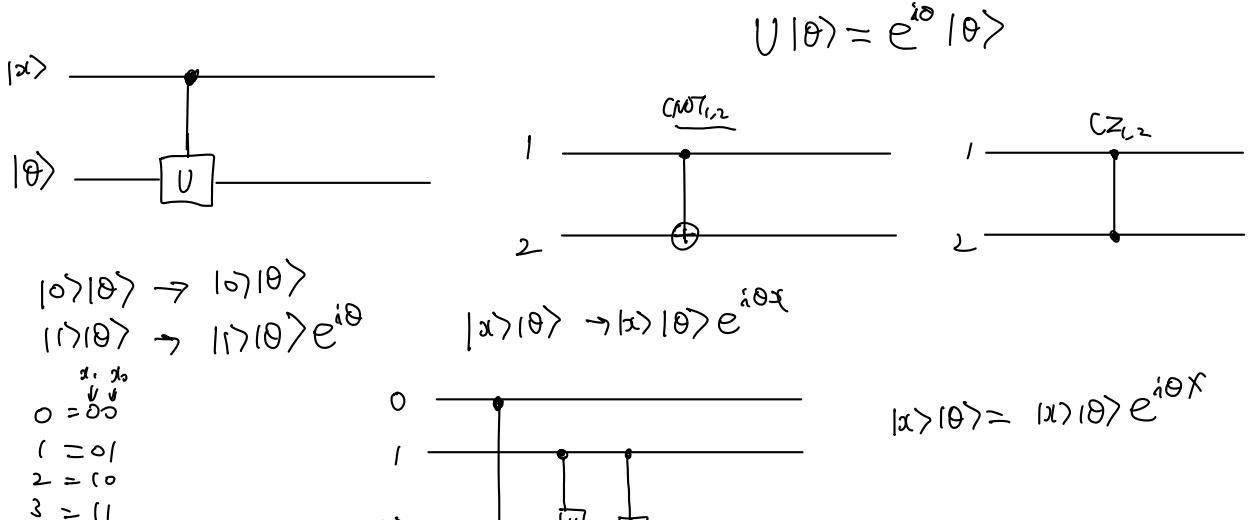
Phase kickback

- More general case: Suppose we have an ability to implement a unitary U. We wish to implement

$$|x\rangle \rightarrow e^{i\theta x}|x\rangle,$$

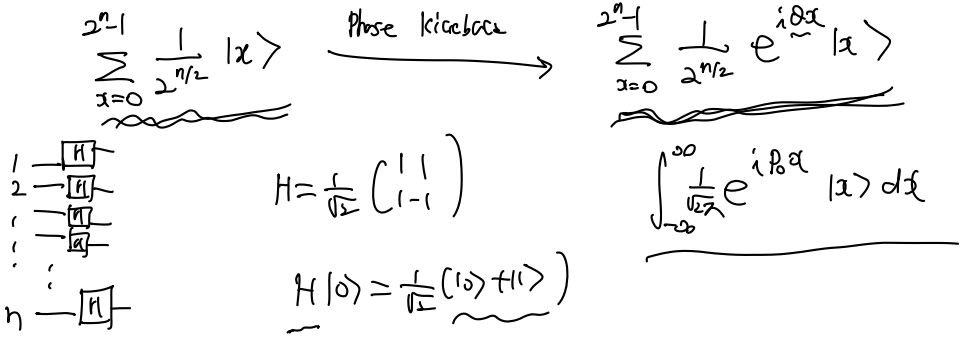
where $e^{i\theta}$ is one of the eigenstates of U.

- How would we do this?



Phase kickback, in superposition

- Instead of beginning with $|x\rangle$, suppose we begin in the $\sum_{x=0}^{2^n-1} \frac{1}{2^{n/2}} |x\rangle$.
- After applying the phase kickback operation, we get a "momentum eigenstate."

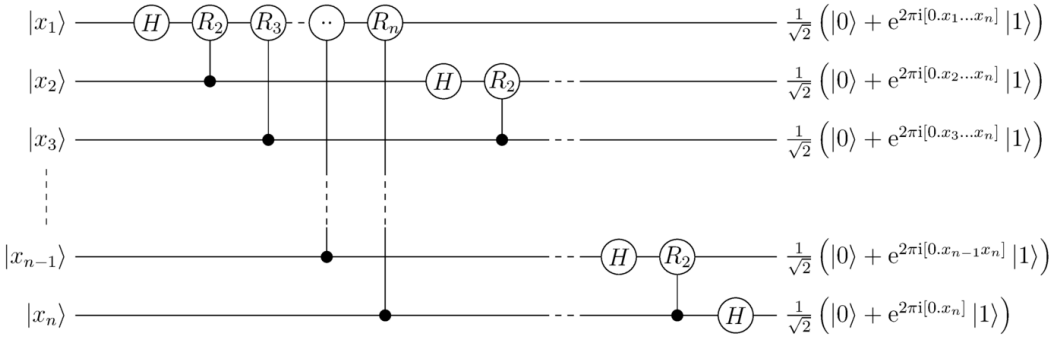


Quantum Fourier Transform

- Quantum Fourier Transform can be performed using $O(n^2)$ one- and two-qubit gates.

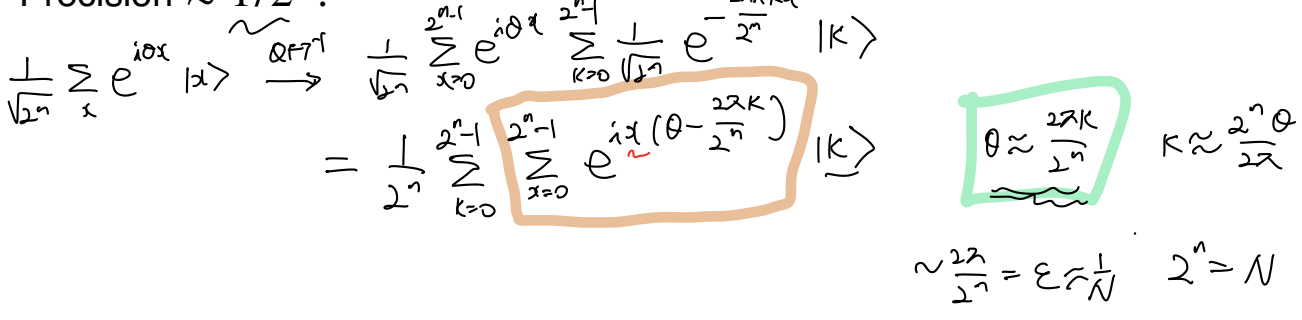
$0, 2^n-1$

$O(n \log n)$



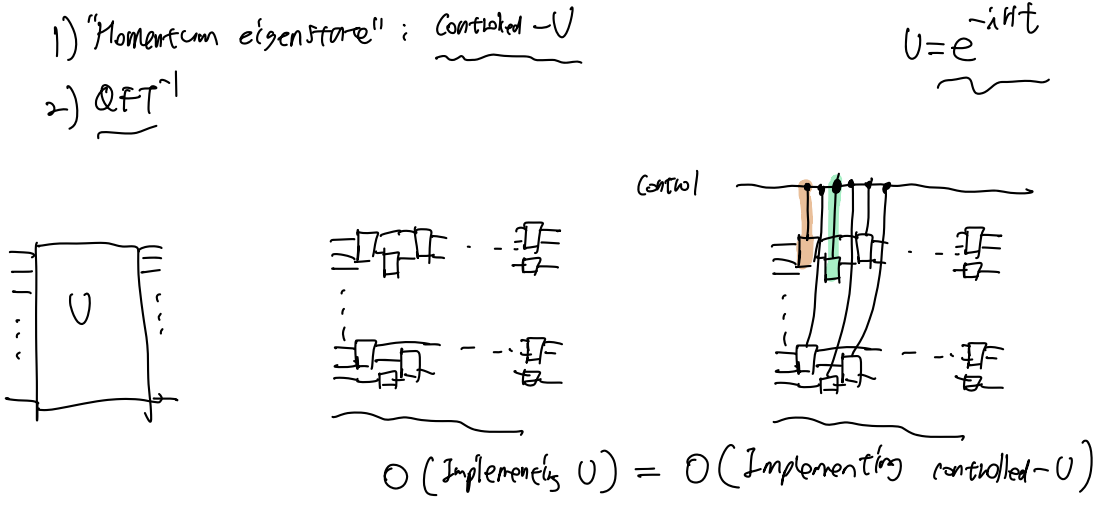
Quantum Phase Estimation in a Nutshell

- In the infinite-precision limit, QPE performs a non-destructive measurement in the eigenbasis of U.
- Procedure: Phase kickback in uniform superposition -> QFT.
- Often the cost of QFT is subleading compared to the implementation of U.
- The dominant cost is 2^n times the cost of implementing controlled-U.
- Precision $\approx 1/2^n$.



Controlled-U

- The cost of implementing controlled-U is comparable to the cost of implementing U.



Applications

$|n\rangle$

- We already established that e^{-iHt} can be implemented efficiently.
- Thus, controlled-e^{-iHt} can be also implemented efficiently.
- Therefore, we can compute the energy of H in time $O(1/\epsilon)$, where ϵ is the precision.
  - This is better than the naive approach of $O(1/\epsilon^2)$.
  - This can lead to a huge difference in quantum chemistry applications.
  - And, there are added benefits. (Next slide)

$H = \sum_i h_i$ $\quad$ $\langle\psi| h_i |\psi\rangle$

Eigenstate assumption

- We assumed that we have access to an eigenstate.
- But didn't we already say that preparing an eigenstate (e.g., ground state) is hard in general?
- More realistically, the initial state will be generally of the following form:
  $|\psi\rangle = \sqrt{1-p_n}\,|n\rangle + \sqrt{p_n}\,|n_\perp\rangle.$

QPE, applied to non-eigenstate

- Suppose we begin with the following state:

$$|\psi\rangle = \sum_n \alpha_n |n\rangle.$$

- If we apply QPE (in the infinite precision limit), we will measure $|n\rangle$ with probability $|\alpha_n|^2$.
- Repeating this many times, you can get a histogram of eigenstates, each labeled by different phases.
- To prepare an eigenstate corresponding to the phase $e^{i\theta_n}$, repeat QPE until you measure θ_n. If you succeed, halt.

But we rarely even know θ_n . . .

- Sometimes we do, e.g., models for which we know the ground state is exactly zero. In that case, we simply repeat measuring θ_n until we get $\theta_n = 0$.
- More generally, we don't know the exact ground state energy. In that case, we simply repeat the measurement many times and pick the smallest θ_n . This will be our "guess" for the ground state energy.
- If the overlap with the true ground state is α, the probability we fail to get the correct ground state energy after m repetition is $(1 - |\alpha|^2)^m$.

$\alpha = \frac{1}{10}$

$m \gg 100$

Application: Ground state preparation

1. Repeat QPE many times to get a good guess on the ground state energy.
2. Repeat QPE many times until you measure your guess of the ground state energy.
3. Done.

- Obviously, this works well only if we can create some state with nonzero overlap with the ground state.

Summary

- QPE is very useful for
  - ❖ energy estimation
  - ❖ ground state preparation
- QPE cost $\approx 1/\epsilon$ times Hamiltonian simulation cost for a unit time.

$\langle \psi | X_1 | \psi \rangle = 0$

$\Pr[X_i = +1] = \frac{1}{2}$

$\Pr[X_i = -1] = \frac{1}{2}$

$\frac{N_+ - N_-}{N} \approx$ $\quad N_+ + N_- = N$

Deviation $\sim \frac{1}{\sqrt{N}} \lesssim \epsilon \quad N \geq \frac{1}{\epsilon^2}$

$H = \sum_{i=1}^{N} \alpha_i P_i$

$H' = \frac{\sum_{i=1}^{N} \alpha_i P_i + \sum_{i=1}^{N} |\alpha_i| I}{2 \sum_{i=1}^{N} |\alpha_i|}$